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Special solutions of the Toda chain and combinatorial numbers

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Abstract
We classify all solutions of the restricted Toda chain from the ansatz that all moments are expressed in terms of polynomials of some variable. We show that each such solution corresponds to the Sheffer class orthogonal polynomials. It is shown that corresponding Hankel determinants are related to some well-known combinatorial numbers.

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1. Introduction

The Toda chain equations [24]

$$\dot{u}_n = u_n(b_n - b_{n-1}) \qquad \dot{b}_n = u_{n+1} - u_n \tag{1.1}$$

with the additional condition

$$u_0 = 0 \tag{1.2}$$

have the well-known relation with the theory of orthogonal polynomials, where the dot indicates the differentiation with respect to t. In what follows we will call equations (1.1) with restriction (1.2) *the restricted Toda chain* (TC) equations.

Let $P_n(x;t)$ be orthogonal polynomials satisfying the three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x) \tag{1.3}$$

with initial conditions

$$P_0 = 1 \qquad P_1(x) = x - b_0. \tag{1.4}$$

We will assume that $u_n \neq 0, n = 1, 2, \ldots$. Then, by the Favard theorem [4], there exists a nondegenerate linear functional σ such that the polynomials $P_n(x)$ are orthogonal with respect to it:

$$\sigma(P_n(x)P_m(x)) = h_n\delta_{nm} \tag{1.5}$$

where h_n are normalization constants. The linear functional σ can be defined through its moments

$$c_n = \sigma(x^n) \qquad n = 0, 1, \ldots. \tag{1.6}$$

It is usually assumed that $c_0 = 1$ (standard normalization condition), but we will not assume this condition in the following. So we will assume that c_0 is an arbitrary nonzero parameter.

Introduce the Hankel determinants

$$D_n = \det(c_{i+j})|_{i,j=0,\ldots,n-1} \qquad D_0 = 1 \quad D_1 = c_0. \tag{1.7}$$

Then the polynomials $P_n(x)$ can be uniquely represented as [4]

$$P_n(x) = \frac{1}{D_n}\begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{1.8}$$

The normalization constants are expressed as

$$h_n = \frac{D_{n+1}}{D_n} \qquad h_0 = D_1 = c_0. \tag{1.9}$$

While the recurrence coefficients u_n satisfy the relation

$$u_n = \frac{h_n}{h_{n-1}} = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{1.10}$$

Thus we have

$$h_n = c_0 u_1 u_2 \cdots u_n. \tag{1.11}$$

Assume now that the polynomials $P_n(x;t)$ depend on a real parameter t through their recurrence coefficients $u_n(t), b_n(t)$, i.e. we will assume that

$$\begin{aligned} &P_{n+1}(x;t) + b_n(t)P_n(x;t) + u_n(t)P_{n-1}(x;t) = xP_n(x;t) \\ &P_0 = 1 \qquad P_1(x;t) = x - b_0(t). \end{aligned} \tag{1.12}$$

Then we have

Theorem 1. *The following statements are equivalent.*

(i) The recurrence coefficients u_n, b_n satisfy the TC equations (1.1) with the restriction $u_0 = 0$ (i.e. $\dot{b}_0 = u_1$).

(ii) The corresponding orthogonal polynomials $P_n(x;t)$ satisfy the relation

$$\dot{P}_n(x;t) = -u_n P_{n-1}(x;t). \tag{1.13}$$

(iii) The moments c_n satisfy the relation

$$\dot{c}_n = c_{n+1} + \frac{\dot{c}_0 - c_1}{c_0}c_n \tag{1.14}$$

where $c_0(t)$ is an arbitrary differentiable function of t.

See [1, 17, 18] for the proof of this theorem.

We note only that it is commonly assumed that $c_0(t) \equiv 1$, but in what follows we will choose another normalization condition

$$\dot{c}_0 = c_1. \tag{1.15}$$

Then the condition (1.14) becomes very simple

$$\dot{c}_n = c_{n+1} \tag{1.16}$$

or, equivalently

$$c_n(t) = \frac{\mathrm{d}^n c_0(t)}{\mathrm{d}t^n}. \tag{1.17}$$

Hence, for the Toda chain case, the Hankel determinants $D_n = D_n(t)$ have the form

$$D_n = \det\left(c_0^{(i+k)}(t)\right)\Big|_{i,k=0,\ldots,n-1} \qquad D_0 = 1 \quad D_1 = c_0 \tag{1.18}$$

where $c_0^{(j)}$ means the jth derivative of $c_0(t)$ with respect to t.

Now we have

Proposition 1. *The restricted TC equations are equivalent also to the equations*

$$\frac{\mathrm{d}^2 \log D_n}{\mathrm{d}t^2} = \frac{D_{n-1}D_{n+1}}{D_n^2} \qquad n = 1, 2, \ldots. \tag{1.19}$$

Proof of this proposition is almost obvious. Equations (1.19) are equivalent to the Hirota bilinear form [8] for the restricted TC equations which was analysed by many authors (see, e.g., [2, 15]). The parametric determinants such as $D_n(t)$ have played a very fundamental role in the Hirota–Sato theory of integrable dynamical systems as tau-functions. As was noticed in [21, 27] relation (1.19) for the Hankel determinants of type (1.18) with (1.16) was firstly obtained by Sylvester and is known today as the Sylvester theorem [11].

Note also that for the Hankel determinants of the form (1.18) we have two useful relations

$$b_n = \frac{\dot{D}_{n+1}}{D_{n+1}} - \frac{\dot{D}_n}{D_n} \tag{1.20}$$

and

$$\dot{h}_n = h_n b_n. \tag{1.21}$$

In particular, for $n = 0$ we have from (1.21)

$$b_0 = \frac{\dot{c}_0}{c_0}. \tag{1.22}$$

Relation (1.22) allows us to restore $c_0(t)$ if the recurrence coefficient $b_0 = b_0(t)$ is known explicitly from Toda chain solutions (1.1).

In this paper we consider a class of explicit solutions of the restricted TC equations through some simple polynomial ansatz for the moments $c_n(t)$. We then show that such solutions correspond to the Sheffer class orthogonal polynomials such as the Meixner, Pollaczek, Laguerre, Charlier and Hermite polynomials. As a by-product, we obtain known expressions for the Hankel determinants of some combinatorial numbers, such as the Euler, the binomial coefficients and Bell numbers.

2. Generating functions of the moments

In the theory of orthogonal polynomials the Stieltjes function $F(z)$ is defined as a generating function of the moments [4]

$$F(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_n}{z^{n+1}} + \cdots. \tag{2.1}$$

If moments c_n depend on t according to the Toda ansatz (1.16), we then have

$$\dot{F}(z;t) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n} + \cdots = zF(z) - c_0. \tag{2.2}$$

In fact, relation (2.2) is equivalent to restricted TC equations (1.16).

We also consider a generating function of another type:

$$\Phi(p) = \sum_{k=0}^{\infty} c_k \frac{p^k}{k!}. \tag{2.3}$$

Note that in number theory for a given sequence of numbers c_n the generating function of type (2.1) is called a G-function, and the generating function of type (2.3) is called an E-function [5]. The relationship between functions $F(z)$ and $\Phi(p)$ is known and is given by the (formal) Laplace transform

$$F(z) = \sum_{k=0}^{\infty} c_k z^{-k-1} = \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{p^k\,\mathrm{e}^{-pz}}{k!}\,\mathrm{d}p = \int_0^{\infty} \mathrm{e}^{-pz}\Phi(p)\,\mathrm{d}p. \tag{2.4}$$

The Stieltjes function $F(z)$ is convenient in many questions connected with the measure for orthogonal polynomials, because in many cases the measure on the real axis can be restored by means of the inverted Stieltjes transform [4]. On the other hand, even for classical polynomials the Stieltjes function $F(z)$ cannot be expressed in terms of elementary functions. Moreover, the convergence domain for the function $\Phi(p)$ is in general larger than that for $F(z)$. So in many questions it is reasonable to deal with the generating function of E-type instead of G-type.

For the case of the restricted TC equations with condition (1.16) we see that the generating function $\Phi(p)$ is given automatically by the formal Taylor expansion

$$\Phi(p;t) = \sum_{k=0}^{\infty} c_k(t)\frac{p^k}{k!} = \sum_{k=0}^{\infty} c_0^{(k)}(t)\frac{p^k}{k!} = c_0(t+p) \tag{2.5}$$

of $c_0(t+p)$. Thus the E-generating function is given just by the shifted $c_0(t+p)$ zero-moment function. The Stieltjes function is given then as the Laplace transform

$$F(z;t) = \int_0^{\infty} \mathrm{e}^{-pz} c_0(t+p)\,\mathrm{d}p. \tag{2.6}$$

It is known [14] that a formal continued fraction expansion of the Laplace transform in terms of a solution of the restricted TC equations is given through the Stieltjes function $F(z;0)$,

$$\int_0^{\infty} \mathrm{e}^{-pz} c_0(p)\,\mathrm{d}p = \frac{c_0(0)\,|}{|\,z+b_0(0)} - \frac{u_1(0)\,|}{|\,z+b_1(0)} - \frac{u_2(0)\,|}{|\,z+b_2(0)} - \cdots. \tag{2.7}$$

Note that if a solution $u_n(t)$, $b_n(t)$ of the restricted TC equations is given, then it is possible to add an arbitrary constant to $b_n(t)$ which leads again to a solution of the same Toda chain:

$$\tilde{b}_n(t) = b_n(t) + \beta \qquad \tilde{u}_n(t) = u_n(t) \tag{2.8}$$

where the constant β does not depend on t. Corresponding orthogonal polynomials $\tilde{P}_n(x;t)$ differ from $P_n(x;t)$ only by shift of the argument:

$$\tilde{P}_n(x) = P_n(x-\beta). \tag{2.9}$$

The function $c_0(t)$ (as follows from (1.22)) changes to

$$\tilde{c}_0(t) = c_0(t)\exp(\beta t). \tag{2.10}$$

Thus the E-generating function for moments of the shifted polynomials $P_n(x-\beta)$ differs only by an exponential multiplier.

Another trivial transformation is the scaling of recurrence coefficients. We have

Proposition 2. *Let $u_n(t)$, $b_n(t)$ be a solution of the restricted TC equations (1.1) and $\gamma \neq 0$ is an arbitrary constant (not depending on t). Then $\gamma^2 u_n(\gamma t)$, $\gamma b_n(\gamma t)$ will also be the solution of the restricted TC equations. The corresponding zeroth moment will be $c_0(\gamma t)$.*

3. Spectral transformations of orthogonal polynomials

Recall that by Christoffel transform [23] we mean new orthogonal polynomials $\tilde{P}_n(x)$ which are obtained from $P_n(x)$ by the formula

$$\tilde{P}_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x-\lambda} \tag{3.1}$$

where λ is an arbitrary parameter such that $P_n(\lambda) \neq 0$, $n = 1, 2, \ldots,$ and

$$A_n = \frac{P_{n+1}(\lambda)}{P_n(\lambda)}. \tag{3.2}$$

The new moments $\tilde{c}_n$ corresponding to orthogonal plynomials $\tilde{P}_n(x)$ are expressed through initial moments by [4, 23]

$$\tilde{c}_n = \mu(c_{n+1} - \lambda c_n) \qquad n = 0, 1, \ldots \tag{3.3}$$

where μ is an arbitrary parameter. It can be easily shown that the Christoffel transformation (3.1) is equivalent to the transformation of the moments (3.3) [4].

Now we consider conditions under which transformation (3.3) is compatible with the restricted TC ansatz (1.16). In general, parameters λ, μ may depend on t. So, we have

$$\dot{\tilde{c}}_n = \dot{\mu}(c_{n+1} - \lambda c_n) + \mu(\dot{c}_{n+1} - \lambda\dot{c}_n - \dot{\lambda}c_n) = \dot{\mu}(c_{n+1} - \lambda c_n) + \mu(c_{n+2} - \lambda c_{n+1}) - \mu\dot{\lambda}c_n.$$

But we should have $\dot{\tilde{c}}_n = \tilde{c}_{n+1}$. The moments c_n are linear independent (because otherwise $D_n = 0$). So we have

$$\dot{\lambda} = 0 \qquad \dot{\mu} = 0. \tag{3.4}$$

Thus the Christoffel transform is compatible with the restricted TC equations if and only if the parameters λ, μ do not depend on t.

It is sufficient to put $\mu = 1$ because $\tilde{c}_0$ is determined up to the constant. We thus have

$$\tilde{c}_n = c_{n+1} - \lambda c_n \qquad n = 0, 1, \ldots. \tag{3.5}$$

For the Hankel determinants $\tilde{D}_n$ constructed from the moments $\tilde{c}_n$ it is not difficult to get

$$\tilde{D}_n = (-1)^n D_n P_n(\lambda). \tag{3.6}$$

4. Special polynomial ansatz and separated variables

In this section we describe concrete examples of the E-generating functions $c_0(t)$ (2.5) which stem from special ansatz for the functional structure of the moments $c_n(t)$. Namely, we assume a separation of variables as follows,

$$c_n(t) = T_n(y(t))c_0(t) \qquad n = 0, 1, \ldots \tag{4.1}$$

where $T_n(y(t))$ is a polynomial of exactly degree n of some (unknown) variable $y(t)$. Note that in [9, 10] some systems of orthogonal polynomials were considered having moments as orthogonal polynomials from some variable. In our approach we *do not* require that $T_n(y)$ be orthogonal polynomials.

The main result is

Theorem 2. *Ansatz (4.1) is compatible with the restricted TC equations if and only if the function $y(t)$ is a solution of the equation*

$$\dot{y}(t) = \sigma(y) \tag{4.2}$$

where $\sigma(y)$ is a (nonzero) polynomial in y with degree less than or equal to 2

$$\sigma(y) = \xi y^2 + \eta y + \zeta \tag{4.3}$$

and the function $\phi(y) = c_0(t(y))$ be a solution of the equation

$$\phi'(y) = \frac{\tau(y)}{\sigma(y)}\phi(y) \tag{4.4}$$

where $\tau(y)$ is a polynomial of exactly first degree,

$$\tau(y) = \alpha y + \beta \tag{4.5}$$

with $\alpha \neq 0$ and β arbitrary parameters, and $t(y)$ is the inverse function with respect to $y(t)$. The restriction between α and ξ

$$\xi \neq -\frac{\alpha}{n} \qquad n = 1, 2, \ldots \tag{4.6}$$

is assumed. The prime in (4.4) indicates the differentiation with respect to y.

Proof. The restricted TC equations are equivalent to the relations $\dot{c}_n = c_{n+1}$. For $n = 0$ we have

$$\dot{c}_0 = T_1(y(t))c_0 \tag{4.7}$$

where $T_1(y) = \alpha y + \beta$ and $\alpha \neq 0$ by condition that $T_1(y)$ is a polynomial of exactly first degree.

For $n = 1$ we have

$$\dot{c}_1 = \alpha\dot{y}c_0 + T_1(y)\dot{c}_0 = \left(\alpha\dot{y} + T_1^2(y)\right)c_0 = T_2(y)c_0 \tag{4.8}$$

whence we derive equation for y

$$\dot{y} = \frac{1}{\alpha}\left(T_2(y) - T_1^2(y)\right) = \sigma(y) \tag{4.9}$$

and we get equation (4.2). By restriction (4.6) we see that $T_2(y)$ is indeed a polynomial of degree exactly 2. Introduce now function $\phi(y) = c_0(t(y))$. This can be done because the inverse function $t(y)$ exists due to condition $\dot{y} \neq 0$. We have from (4.7)

$$\dot{c}_0 = \phi'(y)\dot{y} = c_0 T_1(y).$$

Using (4.9) we arrive at the condition (4.4). Now we have that both $c_1(t)$ and $c_2(t)$ satisfy ansatz (4.1).

Assume, by induction, that $c_n(t) = T_n(y)c_0$ already satisfies this ansatz. For $c_{n+1}(t)$ we have

$$c_{n+1}(t) = \dot{c}_n(t) = \dot{y}T_n'(y)c_0 + T_n(y)\dot{c}_0 = (\sigma(y)T_n'(y) + T_n(y)\tau(y))c_0(t)$$

where we used (4.2) and (4.4). Hence

$$c_{n+1}(t) = T_{n+1}(y)c_0(t)$$

where

$$T_{n+1}(y) = \sigma(y)T_n'(y) + T_n(y)\tau(y).$$

It is easily verified that by restriction (4.6) $T_{n+1}(y)$ is indeed a polynomial of degree exactly $n+1$. This proves the theorem. □

Now we try to recover the functional structure of the recurrence coefficients u_n, b_n of (1.3). We have

$$u_1 = \dot{b}_0 = \frac{\mathrm{d}c_1/c_0}{\mathrm{d}t} = \frac{\mathrm{d}T_1(y(t))}{\mathrm{d}t} = \alpha\sigma(y). \tag{4.10}$$

Then we can find b_1 from

$$b_1 = \frac{\dot{u}_1}{u_1} + b_0 = \tau(y) + \sigma'(y). \tag{4.11}$$

Repeating this process we can find step by step all further coefficients $b_2, u_2, b_3, u_3, \ldots$. We have

Proposition 3. *If moments $c_n(t)$ satisfy the polynomial ansatz (4.1) then the recurrence coefficients $u_n(t), b_n(t)$ have the explicit expressions*

$$\begin{aligned} b_n(t(y)) &= \tau(y) + n\sigma'(y) \\ u_n(t(y)) &= n\sigma(y)\left(\tau'(y) + \tfrac{1}{2}(n-1)\sigma''(y)\right) = n\sigma(y)(\alpha + (n-1)\xi). \end{aligned} \tag{4.12}$$

Proof of this proposition is an elementary application of induction.

We see that the recurrence coefficient $u_n(t)$ has expression with separated variables:

$$u_n(t) = q(t)\kappa_n \qquad n = 0, 1, \ldots \tag{4.13}$$

where $q(t) = \sigma(y(t))$ depends only on t and $\kappa_n = n(\alpha + (n-1)\xi)$ depends only on n.

It is possible to prove an inverse theorem

Theorem 3. *Two ansatzs (4.1) and (4.13) (with the restriction $\kappa_0 = 0$) for solutions of the restricted TC equations are equivalent.*

Such solutions were constructed in [16, 28]. They were also rediscovered in [3]. In [28] and [3] it was established that corresponding polynomials belong to the Sheffer class, namely, Meixner, Pollaczek, Laguerre, Krawtchouk, Charlier and Hermite polynomials. Moreover, it appears that time dynamics for these polynomials is described by Lie group $SU(1,1)$, $SU(2)$ or Heisenberg–Weyl. It is easily established in (4.12) that the compatibility of the ansatz (4.13) with the restricted TC equations [24] leads to the conclusion that κ_n is a polynomial of n of degree 2 or 1 and moreover $\kappa_0 = 0$ and $b_n(t)$ is a linear function of n. Note that $\alpha \neq 0$. If $\deg(\kappa_n) = 2$, we deal with the case of $SU(1,1)$ algebra (infinite chain) or $SU(2)$ algebra (finite chain). Corresponding polynomials are Meixner, Pollaczek, Laguerre ($SU(1,1)$ case) and Krawtchouk ($SU(2)$ case). If $\deg(\kappa_n) = 1$, namely, $\xi = 0$, we deal with the Heisenberg–Weyl algebra. Corresponding polynomials are Charlier and Hermite.

In what follows we describe all these cases separately. We will add arbitrary parameter β to recurrence coefficient $b_n(t)$ to describe as general E-generating functions as possible in accordance with (2.10).

5. Hankel determinants of some combinatorial numbers

We first rewrite the Sylvester relation (1.19), or the bilinear form for the restricted TC equations, as

$$D_{n+1}D_{n-1} = D_n\frac{\mathrm{d}^2 D_n}{\mathrm{d}x^2} - \left(\frac{\mathrm{d}D_n}{\mathrm{d}x}\right)^2 \qquad n = 1, 2, \ldots. \tag{5.1}$$

This implies that the Hankel determinant D_{n+1} of degree $n+1$ is derived from D_{n-1}, D_n and its derivative. Each D_n is uniquely determined from a given differentiable function $c_0(t)$ via the deformation equations of moments (1.16). We consider a separation of variables (4.1) which can be classified according to a choice of the function $\sigma(y) = \xi y^2 + \eta y + \zeta$.

Case 1. $\sigma(y) = 1 - y^2$. In this case $\sigma(y)$ has two distinct real roots. By solving (4.4) we derive

$$\phi(y) = \lambda_1(1-y)^{\mu_1}(1+y)^{\mu_2} \tag{5.2}$$

where $\mu_1 + \mu_2 = -\alpha$, $\mu_1 - \mu_2 = -\beta$ and λ_1 is an arbitrary nonzero constant. By solving (4.2) we have $y(t) = \tanh(t - t_0)$. Therefore we obtain the moment

$$c_0(t) = \phi(y(t)) = \lambda_1 \operatorname{sech}^{-\alpha}(t - t_0)\, \mathrm{e}^{\beta(t-t_0)}. \tag{5.3}$$

The parameter β in the exponential multiplier is a trivial factor of orthogonal polynomials (see (2.10)). The resulting solution of the restricted TC equations is

$$b_n(t) = (\alpha - 2n)\tanh(t - t_0) + \beta \qquad u_n(t) = -n(n - \alpha - 1)\operatorname{sech}^2(t - t_0). \tag{5.4}$$

As was discussed in [3] this solution is related to the Meixner orthogonal polynomials. It also emerges in [13] as a seed solution for constructing a hypergeometric function solution.

Let us choose the parameters in $\tau(y) = \alpha y + \beta$ as $\alpha = -1$, $\beta = 0$ and set $\lambda_1 = 1$ and $t_0 = 0$. Then

$$c_0(t) = \frac{2\mathrm{e}^t}{\mathrm{e}^{2t} + 1}. \tag{5.5}$$

It is to be noted that the moment $c_0(t)$ has an intimate relationship with the Euler numbers E_k, $k = 0, 1, 2, \ldots$, which are the combinatorial numbers defined by

$$E_k = \mathrm{i}^k \sum_{m=0}^{k} 2^m a_m \binom{k}{m} \qquad \frac{2}{\mathrm{e}^t + 1} = \sum_{k=0}^{\infty} a_m \frac{t^m}{m!} \quad \mathrm{i} = \sqrt{-1}. \tag{5.6}$$

Here a_m are rational numbers and $\binom{k}{m}$ are binomial coefficients. The first several Euler numbers are $E_0 = E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 1385$, $E_{10} = 50\,521$ and $E_{2k-1} = 0$. The Euler polynomials $E_k(x)$ of degree k are defined by the following E-generating function:

$$\Phi(p; x) = \frac{2\mathrm{e}^{xp}}{\mathrm{e}^p + 1} = \sum_{k=0}^{\infty} E_k(x)\frac{p^k}{k!} \qquad E_k(x) = \sum_{m=0}^{k} a_m \binom{k}{m} x^{k-m}. \tag{5.7}$$

Thus $c_0(t) = \Phi(2t; 1/2)$ and the coefficients $E_k(1/2)$ of expansion

$$c_0(t) = \sum_{k=0}^{\infty} E_k(1/2)\frac{2^k t^k}{k!} \tag{5.8}$$

give the Euler numbers E_k through the relation $(2\mathrm{i})^k E_k(1/2) = E_k$.

Next we introduce the Hankel determinants $D_n(t) = \det\left(c_0^{(i+j)}(t)\right)\big|_{i,j=0,\dots,n-1}$, $c_0^{(0)}(t) = c_0(t)$, $n = 1, 2, \dots$, defined by the moment (5.5). Let us assume

$$D_n(t) = (-1)^{[n/2]} 2^n \left(\prod_{k=0}^{n-1} k!\right)^2 \frac{\mathrm{e}^{n(n-1)t+nt}}{(\mathrm{e}^{2t}+1)^{n^2}}. \tag{5.9}$$

Inserting

$$D_n \frac{\mathrm{d}^2 D_n}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}D_n}{\mathrm{d}t}\right)^2 = \frac{-n^2\,\mathrm{e}^{2t}}{(\mathrm{e}^{2t}+1)^2} {D_n}^2 \tag{5.10}$$

into the Sylvester relation (5.1) we have

$$\begin{aligned}
\frac{D_{n+1}}{D_n} &= \frac{-n^2\,\mathrm{e}^{2t}}{(\mathrm{e}^{2t}+1)^2} \frac{D_n}{D_{n-1}} \\
&= \frac{-n^2\,\mathrm{e}^{2t}}{(\mathrm{e}^{2t}+1)^2} \frac{-(n-1)^2\,\mathrm{e}^{2t}}{(\mathrm{e}^{2t}+1)^2} \frac{D_{n-1}}{D_{n-2}} \\
&= \frac{(-1)^n (n!)^2\,\mathrm{e}^{2nt}}{(\mathrm{e}^{2t}+1)^{2n}} \frac{D_1}{D_0} \\
&= \frac{(-1)^n (n!)^2 2\mathrm{e}^{2nt+t}}{(\mathrm{e}^{2t}+1)^{2n+1}}.
\end{aligned}$$

Therefore,

$$D_{n+1}(t) = (-1)^{[(n+1)/2]} 2^{n+1} \left(\prod_{k=0}^{n} k!\right)^2 \frac{\mathrm{e}^{n(n+1)t+(n+1)t}}{(\mathrm{e}^{2t}+1)^{(n+1)^2}}.$$

Then (5.9) is proved by induction. It follows from the definition that each $D_n(0)$ gives the Hankel determinant of the Euler polynomials, i.e.

$$D_n(0) = \det(E_{i+j}(x))|_{i,j=0,\dots,n-1} = (-1)^{[n/2]} 2^{n(1-n)} \left(\prod_{k=0}^{n-1} k!\right)^2. \tag{5.11}$$

Let us remark that $D_n(0)$ does not depend on x. Using $(2\mathrm{i})^k E_k(1/2) = E_k$ we have

Proposition 4. *The Hankel determinant of the Euler numbers is given by*

$$\det(E_{i+j})|_{i,j=0,\dots,n-1} = \left(\prod_{k=0}^{n-1} k!\right)^2. \tag{5.12}$$

It is shown here that the Hankel determinant $\det(E_{i+j})|_{i,j=0,\dots,n-1}$ of the Euler numbers is derived through a separation of variables of case 1. Radoux [21] presented the Hankel determinant $\det(E_{2i+2j})|_{i,j=0,\dots,n-1} = \left(\prod_{k=0}^{n-1}(2k)!\right)^2$ of nonzero Euler numbers E_{2i+2j}.

Case 2. $\sigma(y) = -(y-1)^2$. In this case $\sigma(y)$ has a real root of multiplicity 2. A solution of (4.4) is

$$\phi(y) = \lambda_2 (y-1)^{-\alpha} \exp\left(-\frac{\alpha+\beta}{y-1}\right) \tag{5.13}$$

where λ_2 is an arbitrary nonzero constant. By solving (4.2) we have $y(t) = 1/(t-t_0) + 1$. Therefore we obtain the moment

$$c_0(t) = \phi(y(t)) = \lambda_2 (t-t_0)^{\alpha}\,\mathrm{e}^{(\alpha+\beta)(t-t_0)}. \tag{5.14}$$

The corresponding solution of the restricted TC equations is then

$$b_n(t) = \frac{\alpha - 2n}{t - t_0} + \alpha + \beta \qquad u_n(t) = \frac{n(n - \alpha - 1)}{(t - t_0)^2}. \tag{5.15}$$

Let us set $t_0 = -1$, $\alpha + \beta = -x$ and $\lambda_2 = 1$. Then we obtain

$$c_0(t) = (t + 1)^\alpha \, \mathrm{e}^{-xt+x} \tag{5.16}$$

where $(t + 1)^\alpha \, \mathrm{e}^{-xt}$ is a generating function of the Laguerre polynomials $L_k{}^{\alpha-k}(x)$, $(t + 1)^\alpha \, \mathrm{e}^{-xt} = \sum_{k=0}^{\infty} L_k{}^{\alpha-k}(x) t^k$. The condition (4.6) says that $\alpha \neq n, n = 1, 2, \ldots$.

Let us consider the Hankel determinant of the generating function of the generalized binomial coefficients $L_k{}^{\alpha-k}(0)$,

$$\begin{aligned} &D_n(t) = \det(d_{i+j}(t))|_{i,j=0,\ldots,n-1} \qquad D_0(t) = 1 \quad D_1(t) = d_0 \\ &d_0(t) = (t + 1)^\alpha \qquad d_n(t) = \frac{\mathrm{d}^n \, d_0(t)}{\mathrm{d}t^n}. \end{aligned} \tag{5.17}$$

Let us assume

$$D_n(t) = (-1)^{[n/2]} \prod_{k=1}^{n-1} k!(\alpha - k + 1)^{n-k} (t + 1)^{n\alpha - n(n-1)}. \tag{5.18}$$

Inserting

$$D_n \frac{\mathrm{d}^2 D_n}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}D_n}{\mathrm{d}t}\right)^2 = \frac{-n(\alpha - n + 1)}{(t + 1)^2} D_n{}^2 \tag{5.19}$$

into the Sylvester relation (5.1) we have

$$\begin{aligned} \frac{D_{n+1}}{D_n} &= \frac{-n(\alpha - n + 1)}{(t + 1)^2} \frac{D_n}{D_{n-1}} \\ &= \frac{-n(\alpha - n + 1)}{(t + 1)^2} \frac{-(n - 1)(\alpha - n + 2)}{(t + 1)^2} \frac{D_{n-1}}{D_{n-2}} \\ &= \frac{(-1)^n n! \prod_{k=1}^{n} (\alpha - n + k)}{(t + 1)^{2n}} \frac{D_1}{D_0} \\ &= (-1)^n n! \prod_{k=1}^{n} (\alpha - n + k)(t + 1)^{\alpha - 2n}. \end{aligned}$$

Therefore,

$$D_{n+1}(t) = (-1)^{[(n+1)/2]} \prod_{k=1}^{n} k!(\alpha - k + 1)^{n-k+1} (t + 1)^{(n+1)\alpha - (n+1)n}.$$

Then (5.18) is proved by induction. The following proposition is proved.

Proposition 5. *The Hankel determinant of the generalized binomial coefficients $L_k{}^{\alpha-k}(0)$ is given by*

$$D_n(0) = (-1)^{[n/2]} \prod_{k=1}^{n-1} k!(\alpha - k + 1)^{n-k}. \tag{5.20}$$

Case 3. $\sigma(y) = -y^2 - 1$. Here $\sigma(y)$ has two complex conjugate roots. Equation (4.4) is solved to

$$\phi(y) = \lambda_3 (y^2 + 1)^{-\alpha/2} \exp(-\beta \arctan y) \tag{5.21}$$

where λ_3 is an arbitrary nonzero constant. While the corresponding solution of (4.2) is given by $y(t) = \tan(t - t_0)$. Consequently we derive the moment

$$\begin{aligned} c_0(t) &= \phi(y(t)) \\ &= \lambda_3(1 + \tan^2(t - t_0))^{-\alpha/2}\, \mathrm{e}^{-\beta(t-t_0)} \\ &= \lambda_3 \sec^{-\alpha}(t - t_0)\, \mathrm{e}^{-\beta(t-t_0)}. \end{aligned} \tag{5.22}$$

The solution

$$b_n(t) = (\alpha - 2n)\tan(t - t_0) + \beta \qquad u_n(t) = -n(\alpha - n + 1)\sec^2(t - t_0) \tag{5.23}$$

of the restricted TC equations is derived which corresponds to the Pollaczek orthogonal polynomials [3].

It is to be noted that the E-generating function $c_0(t) = \sec t$ of the Euler numbers emerges from (5.22) through a specialization such that $\alpha = -1, \beta = 0$ and $\lambda_3 = 1, t_0 = 0$. Thus the Hankel determinant $\det(E_{i+j})|_{i,j=0,\ldots,n-1}$ of the Euler numbers is derived directly by computing $\lim_{t\to 0} D_n(t)$, where $D_n(t) = \det\left(c_0^{(i+k)}\right)\big|_{i,k=0,\ldots,n-1}$ and $c_0^{(0)} = \sec t$. We omit the proof of it.

Case 4. $\sigma(y) = y + 1$. In this case $\xi = 0$ and $\sigma(y)$ is a linear function of y. The corresponding solution of (4.4) is

$$\phi(y) = \lambda_4\, \mathrm{e}^{\alpha y}(y + 1)^{-\alpha+\beta} \tag{5.24}$$

where λ_4 is a nonzero constant. While equation (4.2) gives $y(t) = \mathrm{e}^{t-t_0} - 1$. Consequently, we have

$$c_0(t) = \phi(y(t)) = \lambda_4\, \exp(\alpha\, \mathrm{e}^{t-t_0} - \alpha - (\alpha - \beta)(t - t_0)). \tag{5.25}$$

The solution of the restricted TC equations is

$$b_n(t) = n\alpha\, \mathrm{e}^{t-t_0} - \alpha + \beta \qquad u_n(t) = n\alpha\, \mathrm{e}^{t-t_0} \tag{5.26}$$

which is related to the Charlier orthogonal polynomials [28].

Let us choose the parameters as $\alpha = 1, \beta = 1$ and set $\lambda_4 = 1$ and $t_0 = 0$. Then an E-generating function

$$c_0(t) = \mathrm{e}^{\mathrm{e}^t - 1} \tag{5.27}$$

of the Bell numbers B_k emerges. Namely

$$c_0(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \qquad B_k = \sum_{m=0}^{k} S(k, m) \tag{5.28}$$

where $S(k, m)$ is the Stirling number of the second kind. Thus $B_k = c_0^{(k)}(t)$. First several Bell numbers are $B_0 = B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203, B_7 = 877$. It is shown [21] that the Hankel determinants $D_n(t) = \det\left(c_0^{(i+j)}(t)\right)\big|_{i,j=0,\ldots,n-1}$ and $D_n(0) = \det(B_{i+j})|_{i,j=0,\ldots,n-1}$ are

$$\begin{aligned} D_n(t) &= \prod_{k=0}^{n-1} k!\, \exp\left(\frac{n(n-1)}{2}t + n\, \mathrm{e}^t - n\right) \\ D_n(0) &= \prod_{k=0}^{n-1} k! \end{aligned} \tag{5.29}$$

respectively. Proof of these formulae is given by using the Sylvester relation (5.1). It is shown that the Hankel determinant $\det(B_{i+j})|_{i,j=0,\ldots,n-1}$ of the Bell numbers is presented through a separation of variables of case 4.

Table 1. Toda chain, combinatorial numbers and orthogonal polynomials.

	Case 1	Case 2	Case 3	Case 4	Case 5
$\sigma(y)$	$-y^2+1$	$-(y-1)^2$	$-y^2-1$	$y+1$	1
$c_0(t)$	$2\mathrm{e}^t/(\mathrm{e}^{2t}+1)$	$(t+1)^\alpha\,\mathrm{e}^{-xt+x}$	$\sec^{-\alpha} t\cdot \mathrm{e}^{-t}$	$\exp(\mathrm{e}^t-1)$	$\exp(-t^2+2xt)$
$u_n(t)$	$-n(n-\alpha-1)\,\mathrm{sech}^2(t)$	$n(n-\alpha-1)/(t+1)^2$	$-n(\alpha-n+1)\sec^2 t$	$n\alpha\,\mathrm{e}^t$	$n\alpha$
Combinatorial number	Euler number E_k	Binomial coeff. $\binom{\alpha}{k}$	Euler number E_k	Bell number B_k	Hermite o.p. $H_k(z)$
Hankel determinant	$\left(\prod_{k=0}^{n-1} k!\right)^2$	$(-1)^{[n/2]}\prod_{k=1}^{n-1} k!(\alpha-k+1)^{n-k}$	$\left(\prod_{k=0}^{n-1} k!\right)^2$	$\prod_{k=0}^{n-1} k!$	$(-2)^{n(n-1)}\prod_{k=0}^{n-1} k!$
Sheffer orthogonal polynomial	Meixner	Laguerre	Pollaczek	Charlier	Hermite

Case 5. $\sigma(y) = 1$. A solution of (4.4) is given by

$$\phi(y) = \lambda_5 \exp\left(\frac{\alpha}{2}y^2 + \beta y\right) \tag{5.30}$$

where λ_5 is a nonzero constant. Equation (4.2) simply gives $y(t) = t - t_0$. We have

$$c_0(t) = \phi(y(t)) = \lambda_5 \exp\left(\frac{\alpha}{2}(t - t_0)^2 + \beta(t - t_0)\right). \tag{5.31}$$

The resulting solution is very simple:

$$b_n(t) = \alpha(t - t_0) + \beta \qquad u_n(t) = n\alpha. \tag{5.32}$$

Let us choose the parameters as $\alpha = -2$, $\beta = 2x$ and set $\lambda_5 = 1$ and $t_0 = 0$. Then an E-generating function

$$c_0(t) = \exp(-t^2 + 2xt) \tag{5.33}$$

of the Hermite polynomials $H_k(x)$ is derived, namely

$$c_0(t) = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}. \tag{5.34}$$

The Hankel determinants $D_n = \det(H_{i+j}(x))|_{i,j=0,\ldots,n-1}$ of Hermite polynomials is found in Radoux [20] by an alternative way. The result is

$$D_n = \det(H_{i+j}(x))|_{i,j=0,\ldots,n-1} = (-2)^{n(n-1)} \prod_{k=0}^{n-1} k!. \tag{5.35}$$

Note that D_n is independent of x. Indeed, the parameter $\beta = 2x$ in (5.33) appears as an exponential multiplier of the moment. We see that the Hankel determinant $\det(H_{i+j}(0))|_{i,j=0,\ldots,n-1}$ of constant terms of the Hermite polynomials is given by using a separation of variables of case 5.

Finally, we would like to mention that all explicit expressions for the Hankel determinants in this section can be obtained directly from the formula

$$D_n = h_0 h_1 \ldots h_{n-1} \tag{5.36}$$

which follows from (1.9). Indeed, normalization coefficients h_n are reconstructed from (1.11) and the recurrence coefficients $u_n(t)$ are known explicitly. Nevertheless, our way of derivation of D_n may be instructive because it exploits only the Toda chain equation (1.19).

Results in this section are summarized in table 1.

6. Concluding remarks

It is known that the Hankel determinant solutions of the restricted TC equations are completely determined by the moment function $c_0(t)$. The key equations are $\dot{c}_n = c_{n+1}, n = 0, 1, \ldots$. In this paper we consider such solutions starting from a polynomial ansatz for the moments $c_n(t) = T_n(y(t))c_0(t)$. The main new result of the paper is the description of all such solutions and their relations with the orthogonal polynomials of the Sheffer class. Another new result is the combinatorial interpretation of these Toda chain solutions. Note that links between Hankel determinants, orthogonal polynomials and some combinatorial numbers were established earlier in [6, 7, 12, 20, 21, 25, 26]. We would like to stress that all these links appear naturally from special solutions of the Toda chain.

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References

[1] Aptekarev A I, Branquinho A and Marcellan F 1997 Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation *J. Comput. Appl. Math.* **78** 139–60

[2] Berezanski Yu M 1986 The integration of semi-infinite Toda chain by means of inverse spectral problem *Rep. Math. Phys.* **24** 21–47

[3] Chen Y and Ismail M E H 1997 Thermodynamic relations of the Hermitian matrix ensembles *J. Phys. A: Math. Gen.* **30** 6633–54

[4] Chihara T 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

[5] Chudnovsky D V and Chudnovsky G V 1993 Hypergeometric and modular function identities, and new rational approximations to and continued fraction expansions of classical constants and functions *A tribute to Emil Grosswald: Number Theory and Related Analysis* (*Contemp. Math.* vol 143) (Providence, RI: American Mathematical Society) pp 117–62

[6] Delsarte Ph 1978 Nombres de Bell et polynômes de Charlier *C. R. Acad. Sci., Paris* **287** A271–A273

[7] Flajolet P 1980 Combinatorial aspects of continued fractions *Discrete Math.* **32** 125–61

[8] Hirota R 1988 Toda molecule equations *Algebraic Analysis* vol 1 (Boston, MA: Academic) pp 203–16

[9] Ismail M E H and Stanton D 1997 Classical orthogonal polynomials as moments *Can. J. Math.* **49** 520–42

[10] Ismail M E H and Stanton D 1998 More orthogonal polynomials as moments *Mathematical Essays in Honor of Gian-Carlo Rota (Cambridge, MA, 1996)* (*Prog. Math.* vol 161) (Boston, MA: Birkhauser) pp 377–96

[11] Karlin S 1968 *Total Positivity* vol 1 (Stanford, CA: Stanford University Press)

[12] Krattenthaler C 1999 Advanced determinant calculus *Séminaire Lotharingien de Combinatoire* **B42q** pp 67

[13] Nakamura A 1998 Explicit N-soliton solutions of the (1 + 1)-dimensional Toda molecule equation *J. Phys. Soc. Japan* **67** 791–8

[14] Nakamura Y 1999 Calculating Laplace transforms in terms of the Toda molecule *SIAM J. Sci. Comput.* **20** 306–17

[15] Nakamura Y and Kodama Y 1995 Moment problem of Hamburger, hierarchies of integrable systems, and the positivity of tau–functions *KdV '95 (Amsterdam, 1995) Acta Appl. Math.* **39** 435–43

[16] Okamoto K 1987 Sur les échelles associées aux fomctions spéciales et l'équation de Toda *J. Fac. Sci. Univ. Tokyo* **34** 709–40

[17] Peherstorfer F 2001 On Toda lattices and orthogonal polynomials *Proc. 5th Int. Symp. Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999) J. Comput. Appl. Math.* **133** 519–34

[18] Peherstorfer F, Spiridonov V and Zhedanov A Toda chain, Stieltjes function and orthogonal polynomials (in preparation)

[19] Penson K A, Blasiak P, Duchamp G, Horzela A and Solomon A I 2004 Hierarchical Dobiński-type relations via substitution and the moment problem *J. Phys. A: Math. Gen.* **37** 3475–87

[20] Radoux C 2000 Addition formulas for polynomials built on classical combinatorial sequences *Proc. 8th Int. Cong. Computational and Applied Mathematics, ICCAM-98 (Leuven) J. Comput. Appl. Math.* **115** 471–7

[21] Radoux C 1979 Calcul effectif de certains déterminants de Hankel *Bull. Soc. Math. Belg. Sér.* B **31** 49–55

[22] Sogo K 1993 Time-dependent orthogonal polynomials and theory of soliton. Applications to matrix model, vertex model and level statistics *J. Phys. Soc. Japan* **62** 1887–94

[23] Szegő G 1975 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)

[24] Toda M 1989 *Theory of Nonlinear Lattices* 2nd edn (*Springer Series in Solid-State Sciences* vol 20) (Berlin: Springer) x+225 pp

[25] Viennot X 1985 A combinatorial theory for general orthogonal polynomials with extensions and applications *Orthogonal Polynomials and Applications (Bar-le-Duc, 1984)* (*Lecture Notes in Math.* vol 1171) (Berlin: Springer) pp 139–57

[26] Wimp J 2000 Hankel determinants of some polynomials arising in combinatorial analysis *Computational Methods from Rational Approximation Theory (Wilrijk, 1999) Numer. Algorithms* **24** 179–93

[27] Witte N S and Bessis D 1999 The Lanczos algorithm for extensive many-body systems in the thermodynamic limit *J. Math. Phys.* **40** 4975–94

[28] Zhedanov A S 1990 Toda lattice: solutions with dynamical symmetry and classical orthogonal polynomials *Teoret. Mat. Fiz.* **82** 11–7 (Russian)
Zhedanov A S 1990 *Theor. Math. Phys.* **82** 6–11 (Engl. Transl.)